

Color Superconductivity, Fermionic Quasinormal Modes in Reissner-Nordström-anti-de Sitter Spacetimes and Supersymmetry

V. K. Oikonomou*

Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22, 04103 Leipzig, Germany

August 15, 2012

Abstract

We study two fermionic systems that have an underlying supersymmetric structure, namely a color superconductor and Dirac fermion in a Reissner-Nordström-anti-de Sitter gravitational background. In the chiral limit of the color superconductor, the localized fermionic zero modes around the vortex form an $N = 2$ with zero central charge $d = 1$ quantum algebra, with all the operators being Fredholm. We compute the Witten index of this algebra and we find an unbroken supersymmetry. The fermionic gravitational system in the chiral limit too, has two underlying unbroken $N = 2$, $d = 1$ supersymmetric algebras. The unbroken supersymmetry in the later is guaranteed by the existence of fermionic quasinormal modes in the Reissner-Nordström-anti-de Sitter background. In this case the operators are not Fredholm and regularized indices are deployed.

Introduction

During the last decade the research towards the interrelation of gravity and condensed matter physical systems has received considerable attention, especially the study of holography in such systems [1–6]. This research stream was further enhanced by the experimental verification of certain condensed matter systems that have topological origin [7], an observation which was done very recently actually [8–10]. Particularly these where time reversal symmetric extensions of the famous topological originating quantum Hall effect in two [11, 12] (topological quantum spin Hall effect) and three dimensions [13–15], known as topological insulators [7]. Of course, the quantum Hall effect is the most famous topologically non trivial state of matter in two dimensions [16]. A non-trivial charge of the single-particle Hamiltonian is intrinsic to a topological state of matter. Along with the topological insulators, the topological superconductors serve as another class of topological states of condensed matter, particularly the $p_x + ip_y$ ones. Moreover gapless

*Vasilis.Oikonomou@mis.mpg.de

localized fermions appear around the vortex core of a vortex defect in topological superconductors. Such theoretical constructions are realized in the surface of three dimensional insulators [7,17]. The most extreme condensed matter states can occur at ultra high quark densities. Of course this is no ordinary matter since the quarks are deconfined, but we are talking about condensed matter physics of QCD. At ultra high densities the QCD coupling is relatively small, a situation that can physically occur in neutron stars for example. In these conditions (low temperature, high density) the quarks may form cooper-like pairs, thus breaking explicitly the gauge symmetry, and forming a so-called color superconductor. Bearing in mind that matter states can be topological in ordinary superconductors, it is natural to ask whether a color superconductor is topological. This questions has been answered and the answer is in the affirmative [7]. In relation to this cooper pair condensation, a color superconductor might be related to a condensation that fermionic fields might experience in evolved AdS gravitational backgrounds. A similar phenomenon to a holographic superconducting phase transition [1–6] was studied in [18], but for an Reissner-Nordström-anti-de Sitter black hole spacetime, and the results where consistent with the holographic superconducting phase transition. Particularly in [18] the order parameter is an Dirac fermion charged through a direct coupling to a Maxwell field. In this sense, this model of a charged Dirac fermion in the background of an Reissner-Nordström-anti-de Sitter black hole, could serve as a simple model of color superconductivity. The approach of the authors of [18] is done by computing the quasinormal modes of the charged Dirac fermion field in the aforementioned curved background. As is well known, quasinormal modes [19–22], describe a long lasting period of damped gravitational waves oscillations. The quasinormal modes are so to speak, the characteristic sound of black holes hence matter field perturbations of such gravitational backgrounds can be a useful tool to study black holes. Moreover the quasinormal modes depend only on a few physical parameters of the black hole, namely, the mass, angular momentum and charge of the black holes, thus rendering the spacetime parameters identification easier. The perturbation of a quantum field in a black hole background consists of three time evolution stages, that is, the wave burst, the quasinormal mode stage and the power-law tail [23]. Owing to the vast number of applications and implications of many theoretical frameworks that embody quasinormal modes, research in this area has attracted a lot of attention. Firstly the existence of a black hole can be directly verified by observing it's fundamental quasinormal mode. Additionally the thermodynamic properties of loop quantum gravity (an appealing alternative to string theory) black holes can be further understood using the quasinormal modes (the imaginary part of the quasinormal modes of the scalar field perturbation is equal to the Barbero-Immirzi parameter [22], see and references therein). Moreover the quasinormal modes of anti de Sitter black holes have a dual physical correspondence to quantities of the dual conformal field theory via the well known AdS/CFT correspondence [29]. From astrophysical aspects, the most interesting spacetimes are the asymptotically flat ones, however the observation of the universe's expansion motivated the study of quasinormal modes in de Sitter and anti de Sitter space times [24]. Quasinormal modes can yield which gravitational systems are stable under dynamical perturbations. Actually a static or non-static solution describing a compact object is stable if all it's quasinormal modes are decaying in time, on the contrary even if one mode is growing, the gravitational system

is unstable [22].

Owing to the fact that the charged Dirac fermion in the background of an Reissner-Nordström-anti-de Sitter black hole could be a simple model of color superconductivity, we present in this paper that in both systems, namely the fermionic spectrum around the boundary vortex of a color superconductor and the fermion in the Reissner-Nordström-anti-de Sitter black hole spacetime, there exists a hidden $N = 2$ supersymmetric quantum mechanics algebra [25–28] (SUSY QM hereafter). Particularly, for the color superconductor system, the supersymmetric algebra occurs for the $m = 0$, $p_z = 0$ (chiral case) case of the Bogoliubov-de Gennes equation. In the case of the fermionic field in the curved gravitational background, the supersymmetric algebra is very closely related to the quasinormal modes spectrum, and the very existence of supersymmetry is guaranteed by the existence of quasinormal modes. In the case of color superconductivity, the supersymmetry is due to the vortex, which actually causes localized fermionic solutions around it. Supersymmetric structures of the same kind around defects were studied in [30, 31] where the case of a superconducting and a cosmic string were analyzed respectively. Supersymmetry in the case where fermionic quasinormal modes are studied in various gravitational backgrounds, was investigated in [32]. In the case we shall exploit in this paper, the fermionic system actually has two $N = 2$ $d = 1$ supersymmetries, the supercharges of which could be related to harmonic superspace extensions [33–41]. The supersymmetric quantum mechanics algebras are very important from physical and mathematical point of views, since these can be directly connected to harmonic superspace [33–41] and to $d = 1$ supersymmetric sigma models with very interesting target space geometries. Furthermore, these supersymmetric extensions provide superextensions of the Landau problem and of the quantum Hall effect [42–44]. In addition $N = 2$ $d = 1$ supersymmetry appears in condensed matter systems, like in graphene for example, see [?]. It is surprisingly interesting that the color superconductors and the fermionic system in Reissner-Nordström-anti-de Sitter black hole (a model that is believed to be the gravitational description of color superconductivity) are linked via the same supersymmetric underlying pattern. However, these supersymmetries are different, owing to the fact that in the case of color superconductors, the operators are Fredholm, while in the case of the gravitational system that is not true.

This paper is organized as follows: In section 1 we exploit the color superconductor model and the underlying $N = 2$ SUSY QM algebra. In section 2 we study the charged Dirac fermion in the background of an Reissner-Nordström-anti-de Sitter black hole and we present the structure of the resulting two SUSY QM. At the end of section 2 we present a global symmetry that the aforementioned fermionic system possess. The conclusions follow thereafter.

1 Superconductors and Vortices

In this section we study the underlying supersymmetry that the fermionic system that describes color superconductivity has. We start with the mean-field model of color super-

conductivity, it's benchmark of which is the Hamiltonian [7]:

$$H_{CSC} = \int dx \left[\psi_{a,f}^\dagger (-i\alpha\partial + \beta m - \mu) \delta_{ab} \delta_{fg} \psi_{b,g} + \frac{1}{2} \psi_{a,f}^\dagger \Delta_{ab,fg}(x) C \gamma^5 \psi_{b,g}^* - \frac{1}{2} \psi_{a,f}^T \Delta_{ab,fg}^\dagger(x) C \gamma^5 \psi_{b,g} \right] \quad (1)$$

with α, β and γ_5 being equal to:

$$\alpha = \gamma^0 \gamma^5 = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}, \quad \beta = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (2)$$

In the above equation (1), the matrix C stands for the charge conjugation matrix, namely $C = i\gamma^2 \gamma^0$, where γ^i are the Dirac gamma matrices. The model that is described by the aforementioned Hamiltonian, contains three colors and three flavors, which are denoted by the letters a, b and f, g respectively, in the Hamiltonian (1). The pairing gap is described by $\Delta_{ab,fg}$, in the Lorentz singlet and even parity channel ($J^p = 0^+$). Its specific dependence on the color and flavor is described by:

$$\Delta_{ab,fg}(x) = \sum_{i=1,2,3} \Delta_i \epsilon_{iab} \epsilon_{ifg} \quad (3)$$

The Hamiltonian (1) after a orthogonal transformation in the color-flavor space, can be brought in the decoupled form, with $H_{SCS} = \sum_i^9 H_j$, with H_j being equal to:

$$\begin{aligned} H_j &= \int d \left[\psi_j^\dagger (-i\mathbf{a}\partial + \beta m - \mu) \psi_j + \frac{1}{2} \psi_j^\dagger \Delta(x)_j C \gamma^5 \psi_j^* - \frac{1}{2} \psi_j^T \Delta^*(x)_j C \gamma^5 \psi_j \right] \\ &= \frac{1}{2} \int d \left(\psi_j^\dagger - \psi_j^T C \gamma^5 \right) \begin{pmatrix} -i\mathbf{a}\partial + \beta m - \mu & \Delta_j(x) \\ \Delta_j^*(x) & i\mathbf{a}\partial - \beta m + \mu \end{pmatrix} \begin{pmatrix} \psi_j \\ C \gamma^5 \psi_j^* \end{pmatrix} \\ &= \frac{1}{2} \int d \Psi_j^\dagger \mathcal{H}_j \Psi_j \end{aligned} \quad (4)$$

The case where $\Delta \neq 0$ describes a fully gapped color flavor locked phase. The Hamiltonian (4) possesses many symmetries, like the charge conjugation symmetry and the time reversal symmetry. Such Hamiltonian have symmetry properties that have been classified and tabulated formally, see for example [46, 47]. The single particle Hamiltonian H has the following charge conjugation symmetry:

$$\mathcal{C}^{-1} H \mathcal{C} = -H^* \quad (5)$$

with \mathcal{C} being equal to:

$$\begin{pmatrix} 0 & -C \gamma^5 \\ C \gamma^5 & 0 \end{pmatrix} \quad (6)$$

Moreover when $\Delta(x)$ is a real number and in addition has a uniform phase over the space, the Hamiltonian has the following transformation properties:

$$T^{-1} H T = H^* \quad (7)$$

where T stands for:

$$T = \begin{pmatrix} \gamma^1 \gamma^3 & 0 \\ 0 & \gamma^1 \gamma^3 \end{pmatrix} \quad (8)$$

It is a common fact in the superconductor literature that for an $2D$ $p_x + ip_y$ superconductor, the non-trivial topological charge of the free space Hamiltonian is closely to a localized fermionic zero mode around a vortex line (see [7] and references therein). Same arguments hold for the Hamiltonian (4). We are interested in zero mode fermionic solutions around vortices, with a non-trivial pairing gap, in the even parity pairing case¹. The theoretical context that underlies the calculation of the fermionic spectrum around a quantized vortex line is pretty much described by the Bogoliubov-de Gennes equation, namely:

$$\begin{pmatrix} -i\mathbf{a}\partial + \beta m - \mu & e^{i\theta}|\Delta(r)| \\ e^{-i\theta}|\Delta(r)| & i\mathbf{a}\partial - \beta m + \mu \end{pmatrix} \Phi(x) = E\Phi(x) \quad (9)$$

where we employed polar coordinates to be our coordinate system. The above Hamiltonian describes the free space one particle Hamiltonian with pairing gap $\Delta(x) = e^{i\theta}|\Delta(r)|$ and under the assumption that the vortex line extents in the z -direction and also that the pairing gap does not depend on z . Additionally, $|\Delta(r)|$ is required to obey $\lim_{r \rightarrow \infty} |\Delta(r)| > 0$, or in words it is required to have a positive non-vanishing asymptotic value. Hence any localized fermion solutions that we will find in the following, can be considered independent of the vortex, a fact that entails some sort of universality of the solutions (see also the comment at end of the present section). The zero modes we shall exploit have a purely topological origin [7] in contrast to other solutions describing bound fermions of Caroli-de Gennes-Matricon type with vortex dependent solutions [7, 48]. The solution to the above equation (9) look like:

$$\Phi(r, \theta, z) = e^{ip_z z} \phi_{p_z}(r, \theta) \quad (10)$$

We shall be mainly interested in the case $m = 0$ and $p_z = 0$, and particularly in the zero energy Bogoliubov-de Gennes equation at $m = p_z = 0$. The last case is the so-called chiral limit, in reference to $m = 0$. The solutions of this equation will actually be the localized zero modes around the vortex line. We can classify the solutions of the zero mode ($E=0$) Bogoliubov-de Gennes equation to left handed and right handed fermion solutions according to their γ_5 parity. These solutions are exponentially localized solutions around the vortex and are equal to [7]:

$$\varphi_R = \frac{e^{i\frac{\pi}{4}}}{\sqrt{\lambda}} \begin{pmatrix} J_0(\mu r) \\ ie^{i\theta} J_1(\mu r) \\ 0 \\ 0 \\ e^{-i\theta} J_1(\mu r) \\ -iJ_0(\mu r) \\ 0 \\ 0 \end{pmatrix} e^{-\int_0^r |\Delta(r')| dr'} \quad (11)$$

¹For more details on the specifics of the superconductors see references [49, 50] and references therein.

in reference to the right handed one, while the left handed one takes the form [7]:

$$\varphi_L = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{\lambda}} \begin{pmatrix} 0 \\ 0 \\ J_0(\mu r) \\ -ie^{i\theta} J_1(\mu r) \\ 0 \\ 0 \\ e^{-i\theta} J_1(\mu r) \\ iJ_0(\mu r) \end{pmatrix} e^{-\int_0^r |\Delta(r')| dr'} \quad (12)$$

The above fermionic system, which is based on the zero modes solutions of the Bogoliubov-de Gennes equation, namely:

$$\begin{pmatrix} -i\mathbf{a}\partial + \beta m - \mu & e^{i\theta} |\Delta(r)| \\ e^{-i\theta} |\Delta(r)| & i\mathbf{a}\partial - \beta m + \mu \end{pmatrix} \Phi(x) = 0 \quad (13)$$

can constitute an $N = 2$ supersymmetric quantum mechanics algebra ($N = 2$ SUSY QM hereafter). To see this, let us briefly exploit the basic features of an unbroken $N = 2$ SUSY QM algebra. The generators of the $N = 2$ algebra are the two supercharges Q_1 and Q_2 and a Hamiltonian H , which obey [25–28],

$$\{Q_1, Q_2\} = 0, \quad H = 2Q_1^2 = 2Q_2^2 = Q_1^2 + Q_2^2 \quad (14)$$

The supercharges can be used to define the new supercharge,

$$Q = \frac{1}{\sqrt{2}}(Q_1 + iQ_2) \quad (15)$$

and the its adjoint,

$$Q^\dagger = \frac{1}{\sqrt{2}}(Q_1 - iQ_2) \quad (16)$$

The new supercharges satisfy,

$$Q^2 = Q^{\dagger 2} = 0 \quad (17)$$

and additionally,

$$\{Q, Q^\dagger\} = H \quad (18)$$

A very important element of the algebra is the Witten parity, W , defined as,

$$[W, H] = 0 \quad (19)$$

which anti-commutes with the supercharges,

$$\{W, Q\} = \{W, Q^\dagger\} = 0 \quad (20)$$

Additionally W satisfies the following,

$$W^2 = 1 \quad (21)$$

The main utility of the Witten parity W , is that it spans the Hilbert space \mathcal{H} of the quantum system to positive and negative Witten parity subspaces, that is,

$$\mathcal{H}^\pm = P^\pm \mathcal{H} = \{|\psi\rangle : W|\psi\rangle = \pm|\psi\rangle\} \quad (22)$$

Hence, the quantum system Hilbert space \mathcal{H} can be written $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. For the present case we shall choose a specific representation for the operators defined above, which for the general case can be represented as:

$$W = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (23)$$

with I the $N \times N$ identity matrix. Recalling that $Q^2 = 0$ and $\{Q, W\} = 0$, the supercharges can take the form,

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \quad (24)$$

and

$$Q^\dagger = \begin{pmatrix} 0 & 0 \\ A^\dagger & 0 \end{pmatrix} \quad (25)$$

Consequently,

$$Q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \quad (26)$$

and also,

$$Q_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -A \\ A^\dagger & 0 \end{pmatrix} \quad (27)$$

The $N \times N$ matrices A and A^\dagger , serve as annihilation and creation operators, with, $A : \mathcal{H}^- \rightarrow \mathcal{H}^+$ and also A^\dagger as, $A^\dagger : \mathcal{H}^+ \rightarrow \mathcal{H}^-$. Based on relations (23), (24), (25) the Hamiltonian H , can take a diagonal form,

$$H = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix} \quad (28)$$

Hence the total supersymmetric Hamiltonian H that describes the supersymmetric system, can be written in terms of the superpartner Hamiltonians,

$$H_+ = A A^\dagger, \quad H_- = A^\dagger A \quad (29)$$

For reasons that will be immediately clear, we define the operator P^\pm , the eigenstates of which, $|\psi^\pm\rangle$, satisfy the following relation:

$$P^\pm |\psi^\pm\rangle = \pm |\psi^\pm\rangle \quad (30)$$

Therefore we call them positive and negative parity eigenstates, parity referring to the P^\pm operator. Representing the Witten operator as in (23), the parity eigenstates can be cast in the following representation,

$$|\psi^+\rangle = \begin{pmatrix} |\phi^+\rangle \\ 0 \end{pmatrix} \quad (31)$$

and also,

$$|\psi^-\rangle = \begin{pmatrix} 0 \\ |\phi^-\rangle \end{pmatrix} \quad (32)$$

with $|\phi^\pm\rangle \in H^\pm$. Using the formalism we just exploited, we construct an $N = 2$ SUSY QM algebra using the fermionic system around the vortex. The Bogoliubov-de Gennes equation can be written as:

$$D\Phi(x) = 0 \quad (33)$$

with D being equal to:

$$D = \begin{pmatrix} -i\mathbf{a}\partial + \beta m - \mu & e^{i\theta}|\Delta(r)| \\ e^{-i\theta}|\Delta(r)| & i\mathbf{a}\partial - \beta m + \mu \end{pmatrix} \quad (34)$$

Based on the above matrix, we can build a supersymmetric algebra. Indeed, the adjoint of D is equal to,

$$D^\dagger = \begin{pmatrix} i\mathbf{a}\partial - \mu & e^{i\theta}|\Delta(r)| \\ e^{-i\theta}|\Delta(r)| & -i\mathbf{a}\partial + \mu \end{pmatrix} \quad (35)$$

The zero modes equation for this matrix is $D^\dagger\Phi'(x) = 0$. The supercharges of the SUSY QM algebra, Q and Q^\dagger can be defined in terms of D and D^\dagger as follows,

$$Q = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & 0 \\ D^\dagger & 0 \end{pmatrix} \quad (36)$$

Moreover, the quantum Hamiltonian of the SUSY QM system is,

$$H = \begin{pmatrix} DD^\dagger & 0 \\ 0 & D^\dagger D \end{pmatrix} \quad (37)$$

It is easy to see that the supercharges (36) and the Hamiltonian and (37), the following relations:

$$\{Q, Q^\dagger\} = H, \quad Q^2 = 0, \quad Q^{\dagger 2} = 0, \quad \{Q, W\} = 0, \quad W^2 = I, \quad [W, H] = 0 \quad (38)$$

But the most interesting feature of this color superconductor related supersymmetric quantum system is that the underlying $N = 2$ supersymmetric quantum mechanical system, has unbroken supersymmetry. Supersymmetry is unbroken for a quantum mechanical system if there exists at least one quantum state in the Hilbert space, $|\psi_0\rangle$, with vanishing energy eigenvalue, that is $H|\psi_0\rangle = 0$. In turn, this entails that $Q|\psi_0\rangle = 0$ and $Q^\dagger|\psi_0\rangle = 0$. For a negative parity state this implies,

$$|\psi_0^-\rangle = \begin{pmatrix} 0 \\ |\phi_0^-\rangle \end{pmatrix} \quad (39)$$

or equivalently $A|\phi_0^-\rangle = 0$. Moreover for a positive parity ground state we have,

$$|\psi_0^+\rangle = \begin{pmatrix} |\phi_0^+\rangle \\ 0 \end{pmatrix} \quad (40)$$

or equivalently $A^\dagger|\phi_0^+\rangle = 0$. Whether supersymmetry is unbroken or not, is very closely related to the number of zero modes of the system. Zero modes are perfectly described by the Witten index. Let n_\pm be the number of zero modes of H_\pm in the subspace \mathcal{H}^\pm . For a finite number of zero modes, n_+ and n_- , we define the Witten index of the system to be,

$$\Delta = n_- - n_+ \quad (41)$$

In the case the Witten index is a non-zero integer, supersymmetry is unbroken for sure. The case for which the Witten index is zero is much more evolved. Indeed, if the Witten index is zero, it and if $n_+ = n_- = 0$ supersymmetry is broken. Conversely, if $n_+ = n_- \neq 0$ the system retains an unbroken supersymmetry. The definition for the Witten index we just gave, holds true for Fredholm operators only. An operator A is Fredholm, if it has discrete spectrum, a fact that is ensured if $\dim \ker A < \infty$. By the same reasoning, if an operator is trace-class, this embodies the Fredholm feature for this operator [26]. Accordingly, the Fredholm index of the operator A is closely related to the Witten index with the former defined as,

$$\text{ind} A = \dim \ker A - \dim \ker A^\dagger = \dim \ker A^\dagger A - \dim \ker A A^\dagger \quad (42)$$

Indeed the relation between the aforementioned two indices is,

$$\Delta = -\text{ind} A = \dim \ker H_- - \dim \ker H_+ \quad (43)$$

As we shall see shortly, the operators D and D^\dagger defined in relations (34) and (35) are Fredholm for the localized solutions (the localization entails specific boundary conditions for the operators which in the end render the operators to be Fredholm) around the vortex. The vector space $\ker D$ is given by the solutions of the equation $D\Phi = 0$, with the solutions Φ being zero at spatial infinity. The last property is equivalent to searching for localized solutions around the vortex. As we have seen earlier, the solutions of the equation $D\Phi = 0$, are given by the solutions of the equation (33), which are the two solutions we found earlier, namely, ϕ_R and ϕ_L and are explicitly given by equations (11) and (12). Hence the two localized solutions constitute the space $\ker D$ for the operator D . In the same line of reasoning, the localized solutions of the equation $D^\dagger\Phi = 0$ are given by:

$$\varphi'_R = \frac{e^{i\frac{\pi}{4}}}{\sqrt{\lambda}} \begin{pmatrix} e^{i\theta} J_1(\mu r) \\ iJ_0(\mu r) \\ 0 \\ 0 \\ J_0(\mu r) \\ -ie^{-i\theta} J_1(\mu r) \\ 0 \\ 0 \end{pmatrix} e^{-\int_0^r |\Delta(r')| dr'} \quad (44)$$

in reference to the right-handed solution, while for the left handed one we have:

$$\varphi'_L = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{\lambda}} \begin{pmatrix} 0 \\ 0 \\ e^{i\theta} J_1(\mu r) \\ -iJ_0(\mu r) \\ 0 \\ 0 \\ J_0(\mu r) \\ ie^{-i\theta} J_1(\mu r) \end{pmatrix} e^{-\int_0^r |\Delta(r')| dr'} \quad (45)$$

In the same line of argument as in the D operator case, the operator D^\dagger is also Fredholm with the two solutions φ'_L and φ'_R constituting the space $\ker D^\dagger$. To make contact with the $N = 2$ SUSY QM algebra, the supercharges are defined in terms of the operators D and D^\dagger and the corresponding zero modes are classified according to their P^\pm parity as follows: The parity odd zero modes are (that is the zero modes of the operator D),

$$|\phi_0^-\rangle_1 = \phi_L \quad |\phi_0^-\rangle_2 = \phi_R \quad (46)$$

while the parity even states are (the zero modes of D^\dagger):

$$|\phi_0^+\rangle_1 = \varphi'_L \quad |\phi_0^+\rangle_2 = \varphi'_R \quad (47)$$

Correspondingly, the zero modes of the Hamiltonian, H are $|\psi_0^+\rangle_1, |\psi_0^+\rangle_2, |\psi_0^-\rangle_1, |\psi_0^-\rangle_2$. Since the two operators D and D^\dagger are Fredholm owing to the finiteness of their corresponding kernels, the Fredholm index of the operator D is given by:

$$\text{ind} D = \dim \ker D - \dim \ker D^\dagger = \dim \ker D^\dagger D - \dim \ker D D^\dagger \quad (48)$$

Hence, the Witten index of the corresponding SUSY QM algebra is given by:

$$\Delta = -\text{ind} D \quad (49)$$

Based on the fact that $\ker D = \ker D^\dagger$ as we found previously, the Witten index of the SUSY QM algebra is zero. Note however that $n_- = n_+ \neq 0$ (using the previously deployed notation) a fact that implies unbroken supersymmetry (for physical systems exploiting similar behavior, that is unbroken SUSY with zero Witten index and other interesting attributes, consult references [51–56]). Let us recapitulate what we found up to now. From the fermionic system around a vortex that is constructed by the zero modes solutions of the Bogoliubov-de Genne equation, we can form an $N = 2$ supersymmetric quantum mechanics algebra with no central charge. The supercharges are constructed by the operators D and D^\dagger which as we proved are Fredholm, in the case the zero mode solutions are localized around the vortex.

1.1 A Brief Comment

Before closing this section, we will address the problem of finding the Witten index in the case we change the pairing gap $\Delta(x)$. For example let the new pairing gap $\Delta(x)'$ be related to the old pairing gap by:

$$\Delta(x)' = \Delta(x) + \Delta_1(x) \quad (50)$$

with $\Delta_1(x) = e^{i\theta}|\Delta_1(r)|$ and $\lim_{r \rightarrow \infty} |\Delta_1(r)| > 0$. At the beginning of this section we mentioned that the localized fermionic solutions around the vortex have some sort of universality, stemming from the fact that the pairing gap does not depend on z . This issue, has its impact on the Witten index, and in fact we shall prove that if we change the pairing gap according to relation (50), the Witten index remains invariant. Hence although the solutions might change, supersymmetry remains unbroken. To see this, we shall make use of a theorem which states that, the Fredholm index of a Fredholm operator D , namely $\text{ind}D$ remains invariant if we add a symmetric odd operator C to this Fredholm operator, that is:

$$\text{ind}(D + C) = \text{ind}D \quad (51)$$

In our case, since the new pairing gap obeys $\lim_{r \rightarrow \infty} |\Delta_1(r)| > 0$, the odd symmetric operator has the following representation:

$$C = \begin{pmatrix} 0 & \Delta_1(x) \\ \Delta_1(x)^* & 0 \end{pmatrix} \quad (52)$$

Hence, the Fredholm index of the operator D , defined in relation (34), is invariant with $\text{ind}(D + C) = \text{ind}D$. Thereby, the Witten index $\Delta = -\text{ind}D$ is also invariant, and hence the same results as in the case corresponding to $\Delta(x)$ hold, that is, supersymmetry is unbroken.

2 $N = 2$ SUSY QM and Massless Dirac Fermion Quasinormal Modes in Reissner-Nordström-anti-de Sitter black hole spacetimes

In this section we shall exploit a system of Dirac fermions in a gravitational background, from which we can construct an $N = 2$ SUSY QM algebra. The gravitational background is that of an Reissner-Nordström-anti-de Sitter black hole spacetime. This background is a potential candidate spacetime that can describe color superconductivity. In view of the AdS/CFT correspondences between gauge theory and gravity, the fact that the aforementioned gravitational system and the color superconductor fermionic system have an underlying $N = 2$ SUSY QM is rather useful. Hence, although the two models are independent at first sight, they have a common underlying symmetry pattern which can be useful.

To be more specific, the supersymmetry we shall present shortly, is very closely related to the quasinormal modes of the Dirac fermionic field in the Reissner-Nordström-anti-de

Sitter black hole background. The perturbation of a black hole can be achieved either by directly perturbing the gravitational background or by simply adding matter or gauge fields in the black hole spacetime [22]. In the linear approximation, the fermionic field has no back-reaction on the metric. The metric in a d -dimensional Reissner-Nordström-anti-de Sitter spacetime is given by:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{d-2,k}^2 \quad (53)$$

where, $f(r)$ is equal to:

$$f(r) = k + \frac{r^2}{L^2} + \frac{Q^2}{4r^{2d-6}} - \left(\frac{r_0}{r}\right)^{d-3} \quad (54)$$

In the above equation, L is the AdS radius, Q is the black hole charge, and r_0 is related to the black hole mass M . The $d\Omega_{d-2,k}^2$ is the metric of constant curvature, with k characterizing the curvature. The value $k > 0$ characterizes the metric of an $d-2$ dimensional sphere, while the $k = 0$ describes R^{d-2} . Finally when $k < 0$ it describes H^{d-2} . We shall focus on the flat case in this paper, since we would like to make contact to a superconductor on a plane. In the 4-dimensional case, the zero curvature Reissner-Nordström-anti-de Sitter metric is,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(dx^2 + dy^2) \quad (55)$$

The corresponding spin connection $\omega_{\hat{a}\hat{b}c}$, is equal to:

$$\omega_{\hat{a}\hat{b}c} = e_{\hat{a}d}\partial_c e_{\hat{b}}^d + e_{\hat{a}d}e_{\hat{b}}^f \Gamma_{fc}^d \quad (56)$$

where, $e_{\hat{a}d}$ denotes the tetrad field, while Γ_{fc}^d denotes the Christoffel connection. The Einstein-Maxwell action for the Dirac fermion field equals to [18]:

$$\begin{aligned} \mathcal{S} = & \frac{1}{2G_4^2} \int d^2x \sqrt{-g} \left(\mathcal{R} - \frac{6}{L^2} \right) \\ & + \mathcal{N} \int d^4x \sqrt{-g} \left(-\frac{1}{4} F_{ab} F^{ab} + i(\bar{\Psi} \Gamma^\alpha (D_a - iqA_a) \Psi - m \bar{\Psi} \Psi) \right) \end{aligned} \quad (57)$$

In the above action (57), G_4 is the 4-dimensional gravitational constant, \mathcal{R} is the corresponding Ricci scalar, \mathcal{N} is a total coefficient characterizing matter fields, and q is the coupling constant between the fermion field and the abelian gauge field A_a . Additionally, the operator D_a is:

$$D_a = \partial_a + \frac{1}{2} \omega_{\hat{c}\hat{b}a} \Sigma^{\hat{c}\hat{b}} \quad (58)$$

with $\Sigma^{\hat{c}\hat{b}} = \frac{1}{4} [\Gamma^{\hat{c}}, \Gamma^{\hat{b}}]$, and the Dirac gamma matrices are related to the vierbeins as, $\Gamma^{\hat{b}} = e_{\hat{a}}^b \Gamma^{\hat{a}}$. A solution of the equations of motion corresponding to the action (57) is:

$$A_t = Q \left(\frac{1}{r} - \frac{1}{r_+} \right), \quad \Psi = 0 \quad (59)$$

In order to extract the quasinormal mode spectrum corresponding to the Reissner-Nordström-anti-de Sitter black hole spacetime, we consider the limit in which the fermionic field does not backreact on the metric and the abelian field, as we also mentioned at the beginning of this section. The wave function solution $\Psi(r, x_\mu)$ can be written in the following form [18]:

$$\Psi(r, x_\mu) = \psi(r) e^{-i\omega t + i\vec{k} \cdot \vec{x}} \quad (60)$$

with $x_\mu = (t, x, y)$ and $\vec{x} = (x, y)$. Using the above form of the function (60), the Dirac equation can be cast into the following form [18]:

$$\sqrt{f} \Gamma^{\hat{r}} \partial_r \psi - \frac{i\omega}{\sqrt{f}} \Gamma^{\hat{t}} \psi + \frac{i\vec{k} \cdot \Gamma^{\hat{\vec{x}}}}{r} \psi + \frac{1}{4} \left(\frac{f'}{\sqrt{f}} + \frac{4\sqrt{f}}{r} \right) \Gamma^{\hat{r}} \psi - (iq\Gamma^\alpha A_\alpha + m) \psi = 0 \quad (61)$$

where $\vec{k} \cdot \Gamma^{\hat{\vec{x}}} = k_x \Gamma^{\hat{x}} + k_y \Gamma^{\hat{y}}$. The Dirac gamma matrices can be written in the following representation:

$$\Gamma^{\hat{t}} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \Gamma^{\hat{i}} = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad (62)$$

with I the identity matrix and σ^i the Pauli matrices, namely:

$$\sigma^{\hat{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{\hat{y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^{\hat{r}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (63)$$

For later convenience, we decompose the fermion field Hilbert space to the chirality operator subspaces, that is:

$$\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} \quad (64)$$

and $P_\pm \Psi = \pm \Psi_\pm$, with $P_\pm = 1 \pm \Gamma^5$ and $\Gamma^5 = i\Gamma^t \Gamma^x \Gamma^y \Gamma^r$. Using the eigenstates Ψ_\pm , the Dirac equations of motion can be cast as [18]:

$$\sqrt{f} \partial_r + \frac{1}{4} \frac{f'}{\sqrt{f}} + \frac{\sqrt{f}}{r} \sigma^{\hat{r}} \psi_- - \frac{i}{r} (\vec{k} \cdot \vec{\sigma}) \psi_- + \frac{i}{\sqrt{f}} (\omega + qA_t) \psi_- - m \psi_+ = 0 \quad (65)$$

and also

$$\sqrt{f} \partial_r + \frac{1}{4} \frac{f'}{\sqrt{f}} + \frac{\sqrt{f}}{r} \sigma^{\hat{r}} \psi_+ - \frac{i}{r} (\vec{k} \cdot \vec{\sigma}) \psi_+ + \frac{i}{\sqrt{f}} (\omega + qA_t) \psi_- - m \psi_- = 0 \quad (66)$$

with $\Psi_+ = \psi_+ e^{-i\omega t + i\vec{k} \cdot \vec{x}}$ and $\Psi_- = \psi_- e^{-i\omega t + i\vec{k} \cdot \vec{x}}$. The set of the above equations (65) and (66) is invariant under the transformation:

$$\omega \rightarrow -\omega, \quad q \rightarrow -q, \quad \psi_+ \rightarrow \psi_- \quad (67)$$

In the rest of this paper we shall be interested in the chiral limit $m = 0$. This will result to an unbroken chiral symmetry for the system, which proves to be very important and could be an underlying link between the fermionic gravitational system and the color

superconductor around a vortex system. We focus on the quasinormal modes of ψ_+ . We set $k_y = 0$. This is because the symmetry that the system possesses on the (\vec{x}, \vec{y}) -plane. Upon rewriting ψ_+ as $\psi_+ = r^{-1} f^{-1/4} \tilde{\psi}$, the equation (66) can be simplified to the following one:

$$\sigma^{\hat{r}} \tilde{\psi}' - \frac{i}{f} (\omega + qA_t - \frac{\sqrt{f}}{r} k_x \sigma^{\hat{x}}) \tilde{\psi} = 0 \quad (68)$$

Decomposing the fermionic field $\tilde{\psi}$, as:

$$\tilde{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (69)$$

the above equation (68), can be recast in the following form:

$$\begin{aligned} \psi_1' - \frac{i}{f} (\omega + qA_t) \psi_1 + \frac{i}{r\sqrt{f}} k_x \psi_2 &= 0 \\ \psi_2' + \frac{i}{f} (\omega + qA_t) \psi_2 - \frac{i}{r\sqrt{f}} k_x \psi_1 &= 0 \end{aligned} \quad (70)$$

The above equations are invariant under the symmetry:

$$\omega \rightarrow -\omega, \quad q \rightarrow -q, \quad k_x \rightarrow -k_x, \quad \psi_1 \rightarrow \psi_2 \quad (71)$$

Using these equations, namely (79) we can construct an $N = 2$ supersymmetric algebra. This algebra is founded on the matrix:

$$D_{RN} = \begin{pmatrix} \partial_r - \frac{i}{f} (\omega + qA_t) & \frac{i}{r\sqrt{f}} k_x \\ -\frac{i}{r\sqrt{f}} k_x & \partial_r + \frac{i}{f} (\omega + qA_t) \end{pmatrix} \quad (72)$$

acting on the vector:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (73)$$

It is obvious that the zero modes of the matrix (81) yield the solutions of equation (79) with respect to ω . But these solutions correspond to the zero modes of the Dirac fermionic system. Therefore the zero mode solutions of the matrix (81) and the quasinormal modes of the Dirac fermionic system are in bijective correspondence. Thereby, the existence of quasinormal modes guarantees the existence of zero modes for the aforementioned matrix. The adjoint of the matrix D_{RN} is equal to:

$$D_{RN}^\dagger = \begin{pmatrix} \partial_r + \frac{i}{f} (\omega^* + qA_t) & \frac{i}{r\sqrt{f}} k_x \\ -\frac{i}{r\sqrt{f}} k_x & \partial_r - \frac{i}{f} (\omega^* + qA_t) \end{pmatrix} \quad (74)$$

Correspondingly, the supercharges of the $N = 2$ algebra Q_{RN} and Q_{RN}^\dagger are equal to:

$$Q_{RN} = \begin{pmatrix} 0 & D_{RN} \\ 0 & 0 \end{pmatrix}, \quad Q_{RN}^\dagger = \begin{pmatrix} 0 & 0 \\ D_{RN}^\dagger & 0 \end{pmatrix} \quad (75)$$

Moreover, the quantum Hamiltonian is equal to,

$$H_{RN} = \begin{pmatrix} D_{RN} D_{RN}^\dagger & 0 \\ 0 & D_{RN}^\dagger D_{RN} \end{pmatrix} \quad (76)$$

The supercharges (75) and the Hamiltonian and (76), satisfy the equations (17) and (19), namely

$$\{Q_{RN}, Q_{RN}^\dagger\} = H, \quad Q_{RN}^2 = 0, \quad Q_{RN}^\dagger{}^2 = 0, \quad \{Q_{RN}, W\} = 0, \quad W^2 = I, \quad [W, H_{RN}] = 0 \quad (77)$$

Hence the algebraic structure of an $N = 2$ SUSY QM algebra, underlies this fermionic system that corresponds to the solution ψ_+ (recall that there is another identical system corresponding to ψ_- , which we describe soon). Let's see if this underlying supersymmetry is broken or unbroken. The last strongly depends on the index of the operator D_{RN} . But since the number of quasinormal modes is a discrete infinite set, and owing to the bijective correspondence between the zero modes of the operator D_{RN} and the quasinormal modes, we conclude that the zero modes form a discrete infinite set. Therefore, the operator D_{RN} is not Fredholm which means that the index of the operator and correspondingly the Witten index must be regularized. In order to do so, we shall make use of the heat-kernel regularized index [25–28], both for the operator D_{RN} , denoted $\text{ind}_t D_{RN}$ and for the Witten index, Δ_t , which are defined as:

$$\begin{aligned} \text{ind}_t D_{RN} &= \text{Tr}(-W e^{-t D_{RN}^\dagger D_{RN}}) = \text{tr}_-(-W e^{-t D_{RN}^\dagger D_{RN}}) - \text{tr}_+(-W e^{-t D_{RN} D_{RN}^\dagger}) \\ \Delta_t &= \lim_{t \rightarrow \infty} \text{ind}_t D_{RN} \end{aligned} \quad (78)$$

The parameter t , is positive number $t > 0$, and moreover the trace tr_\pm , stands for the trace in the subspaces \mathcal{H}^\pm . The heat-kernel regularized index is defined for trace class operators [26]. In the regularized index case, the same hold in reference to supersymmetry breaking, that is if $\Delta_t \neq 0$ supersymmetry is unbroken. When the Witten index is zero, if $\ker D_{RN} = \ker D_{RN}^\dagger = 0$, supersymmetry is broken, while when $\ker D_{RN} = \ker D_{RN}^\dagger \neq 0$ supersymmetry is unbroken. In the case at hand, supersymmetry is unbroken. We can see this without solving the zero mode equation of the D_{RN}^\dagger operator. Indeed the existence of zero modes suffices to argue about supersymmetry. Since $\ker D_{RN} \neq 0$, the zero modes equation for the operator D_{RN}^\dagger can yield two results. Either that $\ker D_{RN}^\dagger \neq 0$ or that $\ker D_{RN}^\dagger = 0$. If the second is true, then the Witten index is different than zero, $\Delta_t \neq 0$, hence supersymmetry is unbroken. In the first case, $\ker D_{RN}^\dagger \neq 0$, it can either be that $\ker D_{RN}^\dagger = \ker D_{RN}$ or that $\ker D_{RN}^\dagger \neq \ker D_{RN}$. In both cases supersymmetry is unbroken. Hence the system that is described by the ψ_+ function has an underlying unbroken $N = 2$ supersymmetry.

Recall that there is another solution to the Dirac equation in this curved background, namely ψ_- . The equations of motion corresponding to ψ_- are equal to:

$$\begin{aligned} \psi'_1 - \frac{i}{f}(-\omega - qA_t)\psi'_1 - \frac{i}{r\sqrt{f}}k_x\psi'_2 &= 0 \\ \psi'_2 + \frac{i}{f}(-\omega - qA_t)\psi'_2 + \frac{i}{r\sqrt{f}}k_x\psi'_1 &= 0 \end{aligned} \quad (79)$$

with,

$$\tilde{\psi}' = \begin{pmatrix} \psi'_1 \\ \psi'_2 \end{pmatrix} \quad (80)$$

and ψ_- being related to $\psi_- = r^{-1}f^{-1/4}\tilde{\psi}'$. By the same reasoning as in the ψ_+ case, the supersymmetric quantum algebra can be built on the matrix:

$$D_{RN'} = \begin{pmatrix} \partial_r - \frac{i}{f}(-\omega - qA_t) & -\frac{i}{r\sqrt{f}}k_x \\ \frac{i}{r\sqrt{f}}k_x & \partial_r + \frac{i}{f}(-\omega - qA_t) \end{pmatrix} \quad (81)$$

acting on the vector:

$$\begin{pmatrix} \psi'_1 \\ \psi'_2 \end{pmatrix} \quad (82)$$

The supercharges of the new algebra are equal to

$$Q_{RN'} = \begin{pmatrix} 0 & D_{RN'} \\ 0 & 0 \end{pmatrix}, \quad Q_{RN'}^\dagger = \begin{pmatrix} 0 & 0 \\ D_{RN'}^\dagger & 0 \end{pmatrix} \quad (83)$$

and the Hamiltonian is

$$H_{RN'} = \begin{pmatrix} D_{RN'} D_{RN'}^\dagger & 0 \\ 0 & D_{RN'}^\dagger D_{RN'} \end{pmatrix} \quad (84)$$

Following the same line of argument as in the previous, we easily find an $N = 2$ underlying supersymmetry. Denoting the algebra corresponding to ψ_- , N_2 and the one corresponding to ψ_+ , N_1 we have come to the result that the Dirac fermionic system in an Reissner-Nordström-anti-de Sitter background, possesses an supersymmetry N , that is the direct sum of two $N = 2$ supersymmetries, namely:

$$N_{total} = N_1 \oplus N_2 \quad (85)$$

The total Hamiltonian of the system is $H_{total} = H_{RN'} + H_{RN}$. It is tempting to investigate if this supersymmetry N_{total} results after the breaking of a larger supersymmetry, for example an $N = 4$ supersymmetry, or even the possibility that a central charge exists. In addition this N_{total} supersymmetry could be the a sign of an underlying higher symmetry (for a quite similar situation but in a different context, consult reference [43] where two $N = 2$, $d = 1$ supersymmetries constitute a $N = 4$ supersymmetry). We defer this investigation to a future publication. Let us just mention that the $N = 4$ supersymmetric algebra is very important in string theory, since extended (with $N = 4, 6, \dots$) supersymmetric quantum mechanics models are the resulting models from the dimensional reduction to one (temporal) dimension of $N = 2$ and $N = 1$ Super-Yang Mills theories [33–41]. In addition, extended supersymmetries serve as superextensions of integrable models like Calogero-Moser systems, Landau-type models [42] and also there exist interesting dualities between various supermultiplets with string theory origin (like T-duality) [58]. But the most salient feature of the extended supersymmetric quantum algebra is that it can

be connected to a generalized harmonic superspace [33–41], with the last being a powerful tool for $N \geq 4$ supersymmetric model building. These harmonic space structures are linked to supersymmetric linear models defined in the target space, for which the harmonic variables give rise to target space harmonics. Note that the Dirac solutions, actually the zero modes of the supercharges, are sections of the total spin bundle [59, 60] over the Riemannian manifold M . Hence we can directly connect these sections to an extended supersymmetric sigma model in harmonic superspace. Although these issues are very interesting both in mathematical and physical aspects, we defer this work to a future article where we address formally all the aforementioned topics.

2.1 A Global $U(1) \times U(1)$

As we saw previously, the fermionic system in the Reissner-Nordström-anti-de Sitter black hole background can constitute a space with two $N = 2$ supersymmetric quantum mechanics algebras, described by the supercharges defined in relations (75) and (83). The two superalgebras are invariant under the transformations:

$$\begin{aligned} Q'_{RN} &= e^{-ia} Q_{RN}, & Q'^{\dagger}_{RN} &= e^{ia} Q^{\dagger}_{RN} \\ Q'_{RN'} &= e^{-ia'} Q_{RN'}, & Q'^{\dagger}_{RN'} &= e^{ia'} Q^{\dagger}_{RN'} \end{aligned} \quad (86)$$

Thus each quantum system is invariant under an R -symmetry of the form of an global- $U(1)$. Correspondingly, the total system is invariant under an $U(1) \times U(1)$ symmetry. Each of the aforementioned $U(1)$ symmetries is a symmetry of the Hilbert states corresponding to the spaces \mathcal{H}_{RN}^{+} , \mathcal{H}_{RN}^{-} and $\mathcal{H}_{RN'}^{+}$, $\mathcal{H}_{RN'}^{-}$. Let, ψ_{RN}^{+} and ψ_{RN}^{-} denote the Hilbert states corresponding to the spaces \mathcal{H}_{RN}^{+} and \mathcal{H}_{RN}^{-} . The $U(1)$ transformation of the states is equal to,

$$\psi'^{+}_{RN} = e^{-i\beta_{+}} \psi^{+}_{RN}, \quad \psi'^{-}_{RN} = e^{-i\beta_{-}} \psi^{-}_{RN} \quad (87)$$

Obviously the parameters β_{+} and β_{-} are global parameters so that $a = \beta_{+} - \beta_{-}$. Accordingly, for the spaces $\mathcal{H}_{RN'}^{+}$, $\mathcal{H}_{RN'}^{-}$ we have,

$$\psi'^{+}_{RN'} = e^{-i\beta'_{+}} \psi^{+}_{RN'}, \quad \psi'^{-}_{RN'} = e^{-i\beta'_{-}} \psi^{-}_{RN'} \quad (88)$$

with $\psi_{RN'}^{+}$ and $\psi_{RN'}^{-}$ the Hilbert states of the spaces $\mathcal{H}_{RN'}^{+}$, $\mathcal{H}_{RN'}^{-}$ respectively. It worths mentioning that in some superconductors, such $U(1)$ symmetries are realized. Particularly an initial $U(1) \times U(1)$ symmetry is broken to a single $U(1)$ (see [57] for details).

Conclusions

In this paper we studied the supersymmetry structure underlying two physical fermionic systems, namely the color superconductor in the chiral limit, around the boundary vortex and a fermionic system in the Reissner-Nordström-anti-de Sitter black hole spacetime. For the first system we found by analyzing the Bogoliubov-de Gennes equation that, in the chiral limit, the localized fermion zero modes of the color superconductor constitute an $N = 2$ supersymmetric quantum mechanics algebra with zero supercharge. Interestingly,

by analyzing the quasinormal modes of the gravitational fermionic system in the Reissner-Nordström-anti-de Sitter background, we found two unbroken $N = 2$ supersymmetries. We stressed the fact that this result is interesting from a mathematical point of view, owing to the fact that we can relate the fermionic gravitational system to a sigma model in harmonic superspace. Note that the unbroken supersymmetry of the system is guaranteed by the very own existence of fermionic quasinormal modes of the gravitational system. This, in turn has its own intrinsic appeal since quasinormal modes depend only on a few physical parameters of the black hole. Since these parameters enter the quantum algebra we have a supersymmetry depending on a few physical parameters and that depends on the existence of quasinormal modes.

The two fermionic systems we exploited in detail are believed to be interrelated, with the fermionic system in curved background being a promising candidate for describing the color superconductor. Thus, the supersymmetries we found, show us that under certain very general assumptions, these two systems have an underlying supersymmetric structure, of $N = 2$, $d = 1$ type. Hence, this common, in some way, underlying theme makes us believe that the two systems might be connected. But this is just an indication and not a direct correspondence between the two models. Additionally the supersymmetries of the two spaces have different Hilbert space structure, a fact that is seen easily from the operators being Fredholm in the color superconductor case, and non Fredholm in the other case. Moreover, the gravitational system has a much more rich structure, and both systems can be related to extended supersymmetric algebras. We hope to further scrutinize these issues in the future.

References

- [1] Steven S. Gubser, Phys.Rev. D 78, 065034 (2008)
- [2] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, Phys. Rev. Lett. 101, 031601 (2008).
- [3] S. A. Hartnoll, Class. Quant. Grav. 26, 224002 (2009) [arXiv:0903.3246 [hep-th]]
- [4] J. M. Maldacena, Adv.Theor. Math. Phys. 2, 231-252 (1998)
- [5] S. Gubser, I.R. Klebanov, and A.M. Polyakov, Phys. Lett. B 428, 105-114 (1998)
- [6] E. Witten, Adv. Theor. Math. Phys. 2, 253-291 (1998)
- [7] Y. Nishida, Phys.Rev.D81, 074004 (2010)
- [8] M. Koenig et al., Science 318, 766 (2007)
- [9] D. Hsieh et al., Nature (London) 452, 970 (2008)
- [10] D. Hsieh et.al., Science 323, 919 (2009)
- [11] C. L. Kane, E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005)

- [12] B. A. Bernevig, S.-C. Zhang, Phys. Rev. Lett. 96, 106802 (2006)
- [13] L. Fu, C. L. Kane, E. J. Mele, Phys. Rev. Lett. 98, 106803 (2007)
- [14] R. Roy, Phys. Rev. B79, 195322 (2009)
- [15] J. E. Moore, L. Balents, Phys. Rev. B75, 121306 (2007)
- [16] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. 49, 405 (1982)
- [17] L. Fu and C. L. Kane, Phys. Rev. Lett. 100, 096407 (2008)
- [18] Rong-Gen Cai, Zhang-Yu Nie, Bin Wang, Hai-Qing Zhang, arXiv:1005.1233
- [19] Kokkotas, K. D., Schmidt, B. G., Living Rev. Rel. 2, 2 (1999)
- [20] Nollert, H. P., Class. Quant. Grav. 16, R159 (1999).
- [21] Berti, E., Cardoso, V. and Starinets, A. O., Class. Quant. Grav. 26, 163001 (2009)
- [22] Konoplya, R. A., Zhidenko, A., Rev. Mod. Phys. 83:793 (2011)
- [23] Yue-Jiang Wu, Zheng Zhao, Phys. Rev. D69, 084015 (2004)
- [24] Jia-Feng Chang, You-Gen Shen, Nucl. Phys. B712, 347 (2005)
- [25] E. Witten, Nucl. Phys. B249, 557 (1985)
- [26] “The Dirac Equation”, Bernd Thaller, Springer 1992
- [27] “Supersymmetric Methods in Quantum and Statistical Physics”, G. Junker, Springer, 1996
- [28] M. Combescure, F. Gieres, M. Kibler, J. Phys. A: Math. Gen. 37, 10385 (2004)
- [29] Kanti, P., Int. J. Mod. Phys. A 19, 4899 (2004)
- [30] V. K. Oikonomou, Gen. Rel.Grav., 44, 1285 (2012)
- [31] V. K. Oikonomou, Mod. Phys. Lett. A25, 2611 (2010)
- [32] V.K. Oikonomou, arXiv: 1204.2395
- [33] F. Delduc, E.A. Ivanov, N = 4 mechanics of general (4, 4, 0) multiplets, Nucl. Phys. B 855 (2012) 815, arXiv:1107.1429
- [34] C.M. Hull, The Geometry of Supersymmetric Quantum Mechanics, hep-th/9910028
- [35] E.A. Ivanov, A.V. Smilga, Dirac Operator on Complex Manifolds and Supersymmetric Quantum Mechanics, arXiv:1012.2069

- [36] A. Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, Hyper-Kaehler metrics and harmonic superspace, *Commun. Math. Phys.* 103 (1986) 515
- [37] F. Delduc, S. Kalitzin, E. Sokatchev, Geometry of sigma models with heterotic supersymmetry, *Class. Quantum Grav.* 7 (1990) 1567
- [38] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E.S. Sokatchev, *Harmonic Superspace*, Cambridge University Press 2001
- [39] E. Ivanov, O. Lechtenfeld, N=4 supersymmetric mechanics in harmonic superspace, *JHEP* 0309 (2003) 073, hep-th/0307111.
- [40] E. Ivanov, O. Lechtenfeld, A. Sutulin, Hierarchy of N=8 Mechanics Models, *Nucl. Phys. B* 790 (2008) 493, arXiv:0705.3064
- [41] F. Delduc, E. Ivanov, arXiv: 1201.3794
- [42] Curtright T., Mezincescu L., Ivanov E., Townsend P.K., Planar super-Landau models revisited, *JHEP*, 020, 25 (2007), hep-th/0612300.
- [43] V. Bychkov, E. Ivanov, *Nucl. Phys. B* 863, 33 (2012)
- [44] Ivanov E., Mezincescu L., Townsend P.K., Planar super-Landau models, *JHEP* 143, 23, (2006), hep-th/0510019
- [45] J. Kailasvuori, *Europhys. Lett.* 87 (2009) 47008
- [46] M. R. Zirnbauer, *J. Math. Phys.* 37, 4986 (1996)
- [47] A. Altland, M. R. Zirnbauer, *Phys. Rev. B* 55, 1142 (1997)
- [48] C. Caroli, P. G. de Gennes, J. Matricon, *Phys. Lett.* 9, 307 (1964)
- [49] M. G. Alford, A. Schmitt, K. Rajagopal, T. Schafer, *Rev. Mod. Phys.* 80, 1455 (2008)
- [50] M. G. Alford, A. Schmitt, K. Rajagopal, and T. Schafer, *Rev. Mod. Phys.* 80, 1455 (2008)
- [51] Francisco Correa, Vit Jakubsky, Luis-Miguel Nieto, Mikhail S. Plyushchay, *Phys. Rev. Lett.* 101, 030403 (2008)
- [52] Francisco Correa, V. Jakubsky, Mikhail S Plyushchay, *J. Phys. A* 41, 485303 (2008)
- [53] Mikhail S. Plyushchay, Adrian Arancibia, Luis-Miguel Nieto, *Phys. Rev. D* 83, 065025 (2011)
- [54] M. S. Plyushchay, *Annals Phys.* 245, 339 (1996)
- [55] Mikhail Plyushchay. *Int. J. Mod. Phys. A* 15, 3679 (2000)
- [56] Francisco Correa Mikhail S. Plyushchay, *Annals Phys.* 322, 2493 (2007)

- [57] K. Odagiri, arXiv:1203.1438
- [58] D. Spector, Int. J. Mod. Phys. A27, 6288 (2005)
- [59] “Geometry, Topology and Physics”, M. Nakahara, Graduate Student Series in Physics, IOP Publishing, 1990
- [60] “Riemannian Geometry and Geometric Analysis”, J. Jost, Universitext, Springer, 2005